Freiburg: Lecture Notes

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Abstract

These are informal notes prepared for a short course of four lectures at the Summer School on Modern Knot Theory (University of Freiburg, 5-10 June 2017). The main topics to be covered are:
(1) Helicity and its role in dynamo theory;
(2) Relaxation under topological constraints;
(3) Minimum energy states of knots and links; and
(4) Topological jumps of minimum area soap-films.

1. Helicity, invariance, and topological interpretation

Topology is concerned with the mathematics of continuous deformation and with properties that remain invariant under continuous deformation.

Continuum mechanics (whether fluid or solid) is by definition also concerned with continuous deformation, both in terms of kinematics (i.e. geometry) and dynamics governed by Newton's laws of motion (together with appropriate relations between stress and (rate of) strain).

There is therefore a natural bridge between these disciplines; this bridge is underpinned by the concept of HELICITY.

1.1. *Some elements of magnetohydrodynamics (MHD)*

We start with some of the elements of MHD, which is concerned with the magnetic field $\mathbf{B}(\mathbf{x}, t)$ in an electrically conducting fluid, and the manner in which it evolves in time; also with the velocity field $\mathbf{u}(\mathbf{x}, t)$, which interacts with the magnetic field through transport and diffusive processes – see below. For

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simplicity, we shall restrict attention to incompressible (volume-preserving) flow: $\nabla \cdot \mathbf{u} = 0$. So both $\mathbf{B}$ and $\mathbf{u}$ are 'solenoidal':

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \cdot \mathbf{u} = 0. \tag{1}$$

The electric current density $\mathbf{j}$ and the vorticity field $\boldsymbol{\omega}$ are given by [1]

$$\mathbf{j} = \nabla \wedge \mathbf{B}, \qquad \boldsymbol{\omega} = \nabla \wedge \mathbf{u}. \tag{2}$$

The electric field $\mathbf{E}(\mathbf{x}, t)$ is related to $\mathbf{B}$ by Faraday's law of induction

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \wedge \mathbf{E}. \tag{3}$$

The electric field in a moving element of fluid (non-relativistic) is $\mathbf{E}' = \mathbf{E} + \mathbf{u} \wedge \mathbf{B}$, and Ohm's Law in a moving medium is

$$\mathbf{j} = \sigma \mathbf{E}' = \sigma(\mathbf{E} + \mathbf{u} \wedge \mathbf{B}), \tag{4}$$

where σ is the conductivity of the medium. We assume this to be constant.
Combining the above leads immediately to the INDUCTION EQUATION

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \wedge (\mathbf{u} \wedge \mathbf{B}) + \eta \nabla^2 \mathbf{B}, \tag{5}$$

where $\eta = \sigma^{-1}$, the magnetic diffusivity of the fluid.

1.2. *Perfect conductivity limit*

If $\sigma = \infty$, i.e. $\eta = 0$, then we have [2]

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \wedge (\mathbf{u} \wedge \mathbf{B}). \tag{6}$$

This equation describes pure transport of the field $\mathbf{B}$ by the flow $\mathbf{u}$. It may be written in an alternative Lagrangian form:

$$\frac{D\mathbf{B}}{Dt} \equiv \frac{\partial \mathbf{B}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{B} = (\mathbf{B} \cdot \nabla)\mathbf{u}, \tag{7}$$

where $D/Dt \equiv \partial/\partial t + \mathbf{u} \cdot \nabla$ is the 'material' or 'Lagrangian' derivative.

The flow $\mathbf{u}$ induces an isotopy:

$$\mathbf{a} \longrightarrow \mathbf{x} = \mathbf{x}(\mathbf{a}, t) \tag{8}$$

where $\mathbf{x}$ is the position at time t of the particle that started from position $\mathbf{a}$ at time $t = 0$.
Eqn. (7) has the solution (essentially due to Cauchy)

$$B_i(\mathbf{x}, t) = B_j(\mathbf{a}, 0) \frac{\partial x_i}{\partial a_j}\,. \tag{9}$$

1. In SI units, $\mu_0 \mathbf{j} = \nabla \wedge \mathbf{B}$, but we ignore the inconvenient factor μ_0, which is irrelevant as far as the mathematical structure is concerned.
2. In pure-mathematical notation, $\nabla \wedge (\mathbf{u} \wedge \mathbf{B}) = \mathcal{L}_{\mathbf{u}}(\mathbf{B})$, the Lie derivative of $\mathbf{B}$ with respect to the flow $\mathbf{u}$.

Here, we use suffix notation, summation over repeated suffices being understood. This equation is the same as that satisfied by a material line element $d\mathbf{x}(\mathbf{a},t)$, and this is enough to demonstrate that the **B**-lines behave just like material lines: i.e. they are 'frozen in the fluid' (Alfvén 1942).

Now $\nabla\cdot\mathbf{B}=0 \Longrightarrow \mathbf{B}=\nabla\wedge\mathbf{A}$, where we may adopt the gauge $\nabla\cdot\mathbf{A}=0$. From (3), we then have

$$\nabla\wedge(\partial\mathbf{A}/\partial t+\mathbf{E})=0, \tag{10}$$

and hence

$$\partial\mathbf{A}/\partial t+\mathbf{E}=-\nabla\phi, \tag{11}$$

for some scalar field ϕ. Equivalently, when $\eta=0$,

$$\partial\mathbf{A}/\partial t=\mathbf{u}\wedge\mathbf{B}-\nabla\phi=\mathbf{u}\wedge(\nabla\wedge\mathbf{A})-\nabla\phi. \tag{12}$$

The Lagrangian form of this equation is

$$\frac{DA_i}{Dt}=u_j\frac{\partial}{\partial x_i}A_j-\frac{\partial\phi}{\partial x_i}\,. \tag{13}$$

1.3. *Helicity and its invariance*

Now let V_L be any bounded material volume of perfectly conducting fluid, with (closed) surface S_L, unit outward normal $\mathbf{n}$. We use the suffix $_L$ to indicate 'Lagrangian', i.e. moving with the flow. Then

$$\mathbf{B}\cdot\mathbf{n}=0\quad\text{at } t=0\quad\Longrightarrow\quad\mathbf{B}\cdot\mathbf{n}=0\quad\text{for all } t>0, \tag{14}$$

by virtue of the frozen-in property. S_L is a 'magnetic surface'.

We define the (magnetic) helicity in V_L by

$$\mathcal{H}=\int_{V_L}\mathbf{A}\cdot\mathbf{B}\,dV, \tag{15}$$

and note immediately that this is gauge-independent (check this!).

Now

$$\begin{aligned}\frac{d\mathcal{H}}{dt}&=\int_{V_L}\left(\frac{D\mathbf{A}}{Dt}\cdot\mathbf{B}+\mathbf{A}\cdot\frac{D\mathbf{B}}{Dt}\right)dV\\&=\int_{V_L}\left[\left(u_j\frac{\partial}{\partial x_i}A_j-\frac{\partial\phi}{\partial x_i}\right)B_i+A_jB_i\frac{\partial}{\partial x_i}u_j\right]dV\quad\text{(from (7) and (13))}\\&=\int_{V_L}B_i\frac{\partial}{\partial x_i}(\mathbf{u}\cdot\mathbf{A}-\phi)\,dV\\&=\int_{S_L}(\mathbf{B}\cdot\mathbf{n})(\mathbf{u}\cdot\mathbf{A}-\phi)\,dS\;=0,\end{aligned} \tag{16}$$

by virtue of (14). Hence, for any such Lagrangian volume V_L, the helicity is constant under all deformations.

Notes:

1. Helicity is a pseudo-scalar quantity: it changes sign under change from right-handed to left-handed frame of reference. It is the simplest measure of the 'chirality' of the field $\mathbf{B}$.

2. The invariance was proved by Woltjer (1958), but under incorrect boundary conditions; I gave the above proof in Moffatt (1969).

3. Helicity has a topological interpretation: for two closed flux tubes of small cross-section carrying fluxes Φ_1 and Φ_2,

$$\mathcal{H} = 2n\Phi_1\Phi_2 \tag{17}$$

where n is the Gauss linking number of the tubes. Exercise: prove this!

4. The helicity of a knotted flux tube with flux Φ is, similarly,

$$\mathcal{H} = h\Phi^2, \quad \text{where } h = Wr + Tw, \tag{18}$$

so invariance of $\mathcal{H}$ coupled with conservation of flux Φ is equivalent to invariance of 'Writhe plus Twist'. The result (18) is more difficult to prove; for this, see Moffatt & Ricca (1992).

5. The helicity may be zero even for chiral field distributions. For example, the Whitehead link is chiral, but has zero linkage, so zero link helicity ($n = 0$ in (17)).

1.4. *Digression on Writhe and Twist*

The writhe of a closed curve C is given by the Gauss-like formula

$$Wr = \frac{1}{4\pi} \int_C \int_C \frac{(\mathbf{x} - \mathbf{x}') \cdot (d\mathbf{x} \wedge d\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}\,. \tag{19}$$

This integral is convergent (despite appearances!), but it is not a topological invariant; it varies continuously under continuous deformation of C. Hence Tw also varies continuously, since $Wr + Tw = \text{cst}$.

The twist Tw is not a property of C alone; it requires consideration of a ribbon with C as one boundary. The twist changes by an integer ± 1 by Dehn surgery: cut the ribbon, turn one end through 2π keeping C fixed, and reconnect.

$$Tw = \mathcal{T} + \mathcal{N}, \quad \text{where } \mathcal{T} = \frac{1}{2\pi} \int_C \tau(s)\, ds, \tag{20}$$

and $\mathcal{N}$ is an integer, the 'internal twist' of the ribbon. $\tau(s)$ is the torsion of C at s, and $\mathcal{T}$ is the 'total torsion' of C.

If C is continuously deformed, it may pass through an 'inflexional configuration' at time $t = t_c$, say – at such an instant, C must contain an inflexion point (i.e. a point at which the curvature c vanishes). The shape of C in the neighbourhood of such a point is a 'twisted cubic' with local parametric representation $\{s, s^2, s^3\}$. At the inflexion point ($s = 0$), the torsion has an integrable singularity, such that $\mathcal{T}$ jumps by ± 1 as t increases through t_c; since Tw is continuous, this jump must be compensated by an equal and opposite jump ∓ 1 in $\mathcal{N}$ – this is exactly how writhe converts to internal twist (or vice versa) in a Reidemeister move I, a phenomenon easily illustrated by manipulation of a flexible belt! During such a move, C must pass through an inflexional configuration.

1.5. *Euler equations*

The Euler equations (Euler 1757) for flow of an ideal (inviscid) fluid are

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}.\nabla)\mathbf{u} = \frac{1}{\rho}\nabla p, \tag{21}$$

where p = pressure, and ρ = density, here assumed constant. Equivalently

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{u} \wedge \boldsymbol{\omega} - \nabla h, \tag{22}$$

where $h = p/\rho + \frac{1}{2}\mathbf{u}^2$, the 'Bernoulli term'. The curl of this is the vorticity equation:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \wedge (\mathbf{u} \wedge \boldsymbol{\omega}), \tag{23}$$

and the analogy with (5) is very clear:

$$\mathbf{u} \Longleftrightarrow \mathbf{A}, \qquad \boldsymbol{\omega} \Longleftrightarrow \mathbf{B}, \qquad \nabla h \Longleftrightarrow \nabla\phi. \tag{24}$$

Helmholtz (1858) showed that, under this Euler evolution, 'vortex lines are frozen in the fluid'. Ten years later Kelvin (Thomson 1867, 1869) recognized that knotted vortex lines would remain permanently knotted, and he based his 'vortex atom theory' on this result.

Just as for the magnetic-field situation, a consequence of this is that the 'kinetic helicity'

$$\mathcal{H} = \int_{V_L} \mathbf{u} \cdot \boldsymbol{\omega}\, dV, \tag{25}$$

must be constant, provided $\boldsymbol{\omega} \cdot \mathbf{n} = 0$ on S_L; and the interpretation of this invariant is that it represents the 'degree of linkage' (or perhaps 'degree of knottedness') of the vortex lines of the flow. [Moreau 1961; Moffatt 1969]

This discovery was actually a big surprise at the time. It was a new integral invariant of the classical Euler equations; it might well have been discovered by Kelvin – it was after all just the sort of result he needed – but was overlooked until, somehow, Moreau stumbled upon it. I discovered it quite independently (I wasn't aware of Moreau's paper until 1978), and only because I had struggled for several years to understand Woltjer's result, which turned out to be closely analogous, despite the fact that (5) is linear, whereas (23), coupled with $\boldsymbol{\omega} = \nabla \wedge \mathbf{u}$, is nonlinear.

Enciso et al. (2016) have recently proved that helicity is the *only* integral invariant of volume-preserving transformations of the Euler equations. So there is *no integral invariant* that can distinguish the Whitehead link from the unlink!

1.6. *The Biot-Savart Law*

If $\boldsymbol{\omega}(\mathbf{x})$ is a localised vorticity field in a fluid of infinite extent, then $\{\boldsymbol{\omega} = \nabla \wedge \mathbf{u},\, \nabla \cdot \mathbf{u} = 0\}$ may be inverted to give the *Biot-Savart law*

$$\mathbf{u}(\mathbf{x}) = \frac{1}{4\pi} \int \frac{\boldsymbol{\omega}(\mathbf{x}') \wedge (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}\, d^3\mathbf{x}'. \tag{26}$$

It is the non-locality of this relationship that makes (23) in general so difficult to solve!

2. Mean field electrodynamics

The following approach dates back to Steenbeck et al. (1966). Let $\mathbf{u}(\mathbf{x}, t)$ be a field of homogeneous turbulence, with $<\mathbf{u}>= 0$. Let l be the scale of the turbulence (i.e. of the 'energy-containing eddies'). We focus on the evolution of a 'seed field' on a scale $L \gg l$.

Split $\mathbf{B}$ into mean and fluctuating parts:

$$\mathbf{B}(\mathbf{x}, t) = \mathbf{B}_0(\mathbf{x}, t) + \mathbf{b}(\mathbf{x}, t), \quad \langle \mathbf{b} \rangle = 0\,. \tag{27}$$

Similarly, split the induction equation (now with $\eta \neq 0$):

$$\partial \mathbf{B}_0/\partial t = \nabla \wedge \boldsymbol{\mathcal{E}} + \eta \nabla^2 \mathbf{B}_0\,, \tag{28}$$

$$\partial \mathbf{b}/\partial t = \nabla \wedge (\mathbf{u} \wedge \mathbf{B}_0) + \nabla \wedge \boldsymbol{\mathcal{F}} + \eta \nabla^2 \mathbf{b}\,, \tag{29}$$

where

$$\boldsymbol{\mathcal{E}} = \langle \mathbf{u} \wedge \mathbf{b} \rangle, \quad \boldsymbol{\mathcal{F}} = \mathbf{u} \wedge \mathbf{b} - \langle \mathbf{u} \wedge \mathbf{b} \rangle\,. \tag{30}$$

Everything is linear (if we regard $\mathbf{u}$ as 'given'; so we expect a linear relationship of the form

$$\mathcal{E}_i = \alpha_{ij} B_{0j} + \beta_{ijk} \frac{\partial B_{0j}}{\partial x_k} + \gamma_{ijkl} \frac{\partial^2 B_{0j}}{\partial x_k \partial x_l} + \dots\,. \tag{31}$$

Simplest situation: isotropic turbulence (statistics invariant under rotation):

$$\alpha_{ij} = \alpha\, \delta ij, \qquad \beta_{ijk} = \beta\, \epsilon_{ijk}, \quad \dots. \tag{32}$$

$$\boldsymbol{\mathcal{E}} = \alpha\, \mathbf{B}_0 - \beta\, \nabla \wedge \mathbf{B}_0 + \gamma\, \nabla \wedge (\nabla \wedge \mathbf{B}_0) + \dots \tag{33}$$

Only the first two terms matter on the scale $L \gg l$.

The first term is the famous $\alpha-$effect [Steenbeck, Krause & Rädler 1966]: mean field parallel to mean magnetic field! The mean-field equation becomes (dropping the suffix $_0$)

$$\frac{\partial \mathbf{B}}{\partial t} = \alpha \nabla \wedge \mathbf{B} + (\eta + \beta) \nabla^2 \mathbf{B} + \dots\,. \tag{34}$$

Notes:

1. α is a pseudo-scalar (like the mean helicity $\mathcal{H} =< \mathbf{u} \cdot \boldsymbol{\omega} >$; it can be non-zero only if the turbulence is 'chiral', i.e. 'lacks reflexional symmetry'. The big problem now is to obtain an explicit expression for α.
2. β is a pure scalar; it simply augments the molecular diffusivity; $\eta_e = \eta + \beta =$'turbulent diffusivity'.

2.1. *The dynamo effect*

Now let $\hat{\mathbf{B}}(\mathbf{x})$ be any field satisfying the 'force-free' condition

$$\nabla \wedge \hat{\mathbf{B}}(\mathbf{x}) = K\,\hat{\mathbf{B}}(\mathbf{x})\,, \tag{35}$$

where K is constant. For such a field,

$$\nabla^2 \hat{\mathbf{B}} = -\nabla \wedge (\nabla \wedge \hat{\mathbf{B}}) = -K^2 \hat{\mathbf{B}}\,, \tag{36}$$

so

$$\frac{\partial \mathbf{B}}{\partial t} = \alpha K\,\mathbf{B} - \eta_e K^2\,\mathbf{B}. \tag{37}$$

So, if $\mathbf{B}(\mathbf{x}, 0) = \hat{\mathbf{B}}(\mathbf{x})$, then

$$\mathbf{B}(\mathbf{x}, t) = \hat{\mathbf{B}}(\mathbf{x})\mathrm{e}^{pt} \tag{38}$$

where

$$p = \alpha K - \eta_e K^2\,. \tag{39}$$

Hence the field grows exponentially in strength (its force-free structure being preserved) provided

$$\alpha K > \eta_e K^2\,, \tag{40}$$

i.e. provided the initial scale of variation of the field $L = |K|^{-1}$ is sufficiently large. So the mean-field approach is self-consistent.

Conclusion: A sufficiently large extent of conducting fluid, in turbulent motion that lacks reflexion symmetry, is generically unstable to the growth of a large-scale magnetic field. Order emerges out of chaos!

2.2. *Relation between α and $\mathcal{H}$*

To calculate α, since (31) must hold for all $\mathbf{B}_0$, we can treat $\mathbf{B}_0$ as constant.
Let

$$\mathbf{u}(\mathbf{x}) = u_0(\sin kz, \cos kz, 0) \qquad \text{and} \quad \mathbf{B}_0 = (0, 0, B_0). \tag{41}$$

Then $\boldsymbol{\omega} = \nabla \wedge \mathbf{u} = ku_0(\sin kz, \cos, kz, 0) = k\,\mathbf{u}$, and the helicity

$$\mathcal{H} = \mathbf{u} \cdot \boldsymbol{\omega} = k\,\mathbf{u}^2 = k\,u_0^2. \tag{42}$$

This is a circularly polarised velocity field of maximal helicity. We then have

$$\begin{aligned} \frac{\partial \mathbf{b}}{\partial t} &= \nabla \wedge (\mathbf{u} \wedge \mathbf{B}_0) + \eta \nabla^2 \mathbf{b} \\ &= (\mathbf{B}_0 \cdot \nabla)\mathbf{u} + \eta \nabla^2 \mathbf{b} \\ &= B_0 \frac{\partial}{\partial z}\mathbf{u} + \eta \nabla^2 \mathbf{b}\,. \end{aligned} \tag{43}$$

We here neglect the 'pain in the neck' term $\boldsymbol{\mathcal{F}} = \mathbf{u} \wedge \mathbf{b} - \langle \mathbf{u} \wedge \mathbf{b} \rangle$; but it is in fact identically zero because it turns out immediately that $\mathbf{u} \wedge \mathbf{b}$ is uniform (so $\mathbf{u} \wedge \mathbf{b} = \langle \mathbf{u} \wedge \mathbf{b} \rangle$).

For, under steady conditions, (43) is easily integrated:

$$\mathbf{b} = \frac{u_0 B_0}{\eta k}(\cos kz, -\sin kz, 0), \quad \text{so } \mathbf{u} \wedge \mathbf{b} = -\frac{u_0^2 B_0}{\eta k}(0, 0, 1)\,, \tag{44}$$

and so

$$\boldsymbol{\mathcal{E}} = <\mathbf{u} \wedge \mathbf{b}> = -\frac{u_0^2 B_0}{\eta k}(0, 0, 1) = -\frac{u_0^2}{\eta k}\,\mathbf{B}\,. \tag{45}$$

Hence apparently, recognising that α is a pseudo-scalar,

$$\alpha = -\frac{u_0^2}{\eta k} = -\frac{1}{\eta k^2}\mathcal{H}. \tag{46}$$

However, the relationship here is in fact non-isotropic. To make it isotropic, instead of (41), we need to consider

$$\mathbf{u}(\mathbf{x}) = u_0(\cos ky + \sin kz,\ \cos kz + \sin kx,\ \cos kx + \sin ky), \tag{47}$$

a special form of the 'ABC-flow'. Now $\mathcal{H} = 3k\, u_0^2$, and

$$\alpha = -\frac{u_0^2}{\eta k} = -\frac{1}{3\eta k^2}\mathcal{H}. \tag{48}$$

Now however the term $\boldsymbol{\mathcal{F}}$ is not identically zero, and the result (48) is correct only in the low conductivity limit when $\boldsymbol{\mathcal{F}}$ can be neglected. (This is the 'first-order-smoothing approximation', analogous to the Born approximation in quantum mechanics.)

For homogeneous turbulence, (48) generalises to

$$\alpha = -\frac{1}{3\eta}\int k^{-2}\hat{\mathcal{H}}(k)\, dk\,, \tag{49}$$

where $\hat{\mathcal{H}}(k)$ is the 'helicity spectrum function'.

3. Magnetic Relaxation

This process can be considered the counterpart of the dynamo process. We suppose that at time $t = 0$ we have a magnetic field $\mathbf{B}(\mathbf{x}, 0) = \mathbf{B}_0$ in a perfectly conducting fluid at rest: $\mathbf{u}(\mathbf{x}, 0) = 0$.

The field $\mathbf{B}_0$ satisfies $\nabla \cdot \mathbf{B}_0 = 0$, is assumed 'smooth' (i.e. continuous and with continuous derivatives up to any order, as required by the context) but otherwise has arbitrary topological complexity; for example it may consist of a knotted flux tube, or of several knotted tubes linked in an arbitrary manner. Or the $\mathbf{B}$-lines may be chaotic. We suppose that the fluid is contained in a finite domain $\mathcal{D}$, with $\mathbf{n} \cdot \mathbf{B}_0 = 0$ on the surface $\partial\mathcal{D}$.

So what happens? The Lorentz force per unit volume in the fluid is

$$\mathbf{F} = \mathbf{j} \wedge \mathbf{B}, \tag{50}$$

and in general this is rotational at $t = 0$, i.e.

$$\nabla \wedge (\mathbf{j}_0 \wedge \mathbf{B}_0) \neq 0, \tag{51}$$

so it can't be balanced simply by a pressure distribution. The fluid must move in response to this force, and the velocity $\mathbf{u}(\mathbf{x}, t)$ then satisfies the Navier-Stokes equation, augmented by this Lorentz force:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \mathbf{j}\wedge\mathbf{B} + \mu\nabla^2\mathbf{u}\,, \tag{52}$$

where ρ is the fluid density and μ its viscosity, both assumed uniform.

As the fluid moves, it transports the magnetic field with it, according to (6) – remember, we are assuming perfect conductivity. This is a mathematical artifice, but the end justifies the means [3].

At the same time, energy is dissipated by viscosity: let

$$M(t) = \tfrac{1}{2}\int_{\mathcal{D}} \mathbf{B}^2\, dV \; = \text{magnetic energy} \qquad E(t) = \tfrac{1}{2}\int_{\mathcal{D}} \rho\mathbf{u}^2\, dV \; = \text{kinetic energy}\,. \tag{53}$$

Then

$$\begin{aligned} \frac{dM}{dt} &= \int_{\mathcal{D}} \mathbf{B}\cdot\frac{\partial \mathbf{B}}{\partial t}\, dV \\ &= \int_{\mathcal{D}} \mathbf{B}\cdot\nabla\wedge(\mathbf{u}\wedge\mathbf{B})\, dV \\ &= \int_{\mathcal{D}} \mathbf{j}\cdot(\mathbf{u}\wedge\mathbf{B})\, dV = -\int_{\mathcal{D}} \mathbf{u}\cdot(\mathbf{j}\wedge\mathbf{B})\, dV\,, \end{aligned} \tag{54}$$

and, using the no-slip condition $\mathbf{u} = 0$ on $\partial\mathcal{D}$,

$$\begin{aligned} \frac{dE}{dt} &= \int_{\mathcal{D}} \left[-(\mathbf{u}\cdot\nabla)(\tfrac{1}{2}\mathbf{u}^2 + p/\rho) + \mathbf{u}\cdot(\mathbf{j}\wedge\mathbf{B}) - \mu\mathbf{u}\cdot\nabla\wedge\boldsymbol{\omega}\right]\, dV \\ &= \int_{\mathcal{D}} \left[\mathbf{u}\cdot(\mathbf{j}\wedge\mathbf{B}) - \mu\boldsymbol{\omega}^2\right]\, dV\,. \end{aligned} \tag{55}$$

So

$$\frac{d}{dt}\left(M(t) + E(t)\right) = -\mu\int_{\mathcal{D}} \boldsymbol{\omega}^2\, dV\,, \tag{56}$$

so the magnetic energy is converted to kinetic energy through the term $-\int_{\mathcal{D}} \mathbf{u}\cdot(\mathbf{j}\wedge\mathbf{B})\, dV$ and dissipated by viscosity for so long as $\int_{\mathcal{D}} \boldsymbol{\omega}^2\, dV \neq 0$.

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3. This phrase is commonly attributed to Macchiavelli; what he actually said was this: "For that reason, let a prince have the credit of conquering and holding his state, the means will always be considered honest, and he will be praised by everybody because the vulgar are always taken by what a thing seems to be and by what comes of it; and in the world there are only the vulgar, ..." He also said: "The first method for estimating the intelligence of a ruler is to look at the men he has around him." Relevant to USA today?

$M(t)$ decreases if $\mathbf{u}$ is, on average, parallel to $\mathbf{j}\wedge\mathbf{B}$, thus responding in a natural way to the Lorentz force.

[This is just the opposite of dynamo action which requires that $\mathbf{u}$ should be, on average, *anti-parallel* to $\mathbf{j}\wedge\mathbf{B}$, thus working *against* the Lorentz force; this requires other sources of energy, usually gravitational energy deriving from buoyancy forces.]

3.1. *Lower bound on the magnetic energy*

The fact that the energy decreases is not, on its own, very useful; after all, it could conceivably decrease to zero, with then $\mathbf{B} \equiv 0$ and $\mathbf{u} \equiv 0$ and we would have learnt nothing. Here, however, the conserved helicity comes to our assistance:

We have first the Schwarz inequality:

$$|\mathcal{H}| \leq \left(\int_{\mathcal{D}} \mathbf{A}^2 \, dV \int_{\mathcal{D}} \mathbf{B}^2 \, dV \right)^{1/2} , \qquad (57)$$

where $\mathcal{H}$ is defined for the whole domain $\mathcal{D}$. Secondly, we have a Poincaré inequality:

$$R := \frac{\int_{\mathcal{D}} \mathbf{B}^2 \, dV}{\int_{\mathcal{D}} \mathbf{A}^2 \, dV} \geq q_0^{-2}, \qquad (58)$$

where q_0 is a constant with the dimensions of [length]; this inequality comes from minimising the 'Rayleigh quotient' R, by standard methods of the Calculus of Variations; the length q_0 is then determined, via a Helmholtz problem, by the size and shape of the domain $\mathcal{D}$ [4].

We now combine these two inequalities to give

$$\int_{\mathcal{D}} \mathbf{B}^2 \, dV \geq q_0^{-1} |\mathcal{H}| \, . \qquad (59)$$

This inequality was first obtained by Arnold (1974) (in the most obscure Russian reference ever), so we should describe it as Arnold's inequality! The paper was republished in English translation in the more accessible *Selnik* (1986).

Anyway, this gives us a lower bound on $M(t)$ provided $\mathcal{H} \neq 0$.

[Freedman (1988) showed that a positive lower bound exists even if $\mathcal{H} = 0$, provided simply that the topology of the $\mathbf{B}$-field is nontrivial (i.e.there exist $\mathbf{B}$-lines that can't be shrunk to a point in $\mathcal{D}$ without trapping other $\mathbf{B}$-lines); the Whitehead link, or the Borromean rings, provide good examples!]

3.2. *Nontrivial limit state*

OK, so now we have $M(t) + E(t)$ monotonic decreasing and with a positive lower bound; so it tends to a constant as $t \to \infty$. It follows from (56) that

4. If the fluid is of infinite extent, then q_0 is presumably less than the maximum span of the magnetic field distribution during the relaxation process; more precision needed here – and proof!

$$\int_{\mathcal{D}} \boldsymbol{\omega}^2 \, dV \to 0 \,, \tag{60}$$

and so (Poincaré again)

$$\int_{\mathcal{D}} \mathbf{u}^2 \, dV < q_0^2 \int_{\mathcal{D}} \boldsymbol{\omega}^2 \, dV \to 0 \text{ also.} \tag{61}$$

Hence, *with the plausible assumption that no nasty singularities of* $\mathbf{u}$ *appear in the decay process* (and this still requires proof), it follows that

$$\mathbf{u} \to 0 \quad \text{everywhere in } \mathcal{D}. \tag{62}$$

So now, going back to (52), in the limit $\mathbf{u} \equiv 0$, we have

$$\mathbf{j}^E \wedge \mathbf{B}^E = \nabla p^E \,, \quad \mathbf{j}^E = \nabla \wedge \mathbf{B}^E \,, \tag{63}$$

(where we use the superfiix E for 'equilibrium'), i.e. a state of magnetostatic equilibrium in which the Lorentz force is everywhere balanced by the pressure gradient in the fluid.

So we started with a field $\mathbf{B}_0(\mathbf{x})$ of arbitrary topology. The topology of this field is conserved throughout the decay process because of the frozen-in property; and we end up with a magnetostatic field with the same topology. Well, not quite, because in the limit $t \to \infty$, discontinuities of $\mathbf{B}$ can (and do) appear; but still, reconnections of $\mathbf{B}$-lines cannot occur. In this sense, the limit state is 'topologically accessible' from the initial state. Also, the energy of the field is obviously minimised in this limit state.

So, in summary, it would appear that, given a field $\mathbf{B}_0(\mathbf{x})$ in $\mathcal{D}$ of arbitrary topology, there exists a minimum-energy magnetostatic field $\mathbf{B}^E(\mathbf{x})$ that is topologically accessible from $\mathbf{B}_0(\mathbf{x})$.

3.3. *The second Euler analogy*

The magnetostatic equation (64) may be compared with the equation for *steady* Euler flow

$$\mathbf{u} \wedge \boldsymbol{\omega} = \nabla h \,, \quad \boldsymbol{\omega} = \nabla \wedge \mathbf{u}. \tag{64}$$

Here there is a second analogy, quite distinct from (24):

$$\mathbf{u} \Longleftrightarrow \mathbf{B}, \qquad \boldsymbol{\omega} \Longleftrightarrow \mathbf{j}, \qquad \nabla h \Longleftrightarrow -\nabla p \,. \tag{65}$$

Note the minus sign!

By means of this analogy, we can simply translate the above result:

Given a field $\mathbf{U}_0(\mathbf{x})$ in $\mathcal{D}$ of arbitrary topology, there exists a steady Euler flow $\mathbf{u}^E(\mathbf{x})$ that is topologically accessible from $\mathbf{U}_0(\mathbf{x})$.

Notes:

1. It is the streamline topology that can be prescribed here, not the vortex line topology; great care is needed in the interpretation of these analogies! Note also that we cannot assert that the 'steady Euler flow' is one of minimum energy; this is because perturbations of this flow must be 'isovortical' – i.e. with frozen-in vorticity – whereas perturbations of the magnetostatic equilibrium are at frozen-in $\mathbf{B}$.

2. The analogous Euler flows are in fact generically unstable; this is just the difficulty encountered by Kelvin. If only he had framed his theory in terms of magnetic flux tubes rather than vortex tubes, it might have survived much longer!

3. The theory has had a renaissance in the exotic context of the quark-gluon plasma, as described in Lou Kauffman's open lecture.

4. Enciso & Peralta-Salas (2015) have proved the existence of arbitrarily knotted vortex tubes in steady Euler flows, as predicted by the above heuristic argument. The velocity field $\mathbf{u}(\mathbf{x})$ in these flows are however $\mathrm{O}(r^{-1})$ at infinity, so have infinite energy.

3.4. *Application to knots and links (Moffatt 1990)*

Let K be an arbitrary knot, and let the initial field $\mathbf{B}_0$ be confined to a tube T centred on K. Let V be the volume of the tube; this is conserved under volume-preserving flow (or isotopy, if you prefer). Let Φ be the flux of $\mathbf{B}_0$ along T; this is also conserved under frozen-field distortion. The invariant helicity is given by $\mathcal{H} = h\Phi^2$.

We let the field relax to a minimum energy state $\mathbf{B}^E$ as described above. From a physical point of view, this relaxation occurs because the $\mathbf{B}$-lines contract due to the Maxwell tension in them (just as a stretched elastic band contracts if released from rest). So the tube length decreases, and the cross-section must increase so that V remains constant. This contraction is impeded when the tube makes contact with itself, and adjustment then continues until the minimum energy state is attained.

The energy M^E in this state is determined by V and Φ and the dimensionless helicity h. So on purely dimensional grounds, this minimal energy is given by [5]

$$M^E = m(h)\Phi^2 V^{-1/3}\,, \tag{66}$$

where $m(h)$ is a dimensionless function, an intrinsic property of the knot K.

But, as previously indicated, we should allow for non-uniqueness of the minimum energy states; in this case, we order them:

$$0 < m_0(h) \le m_1(h) \le m_2(h) \le \ldots\,, \tag{67}$$

and we may talk of the 'energy spectrum' $\{m_i(h)\}$ of the knot K (there may be degeneracies; hence $\le$ rather than $<$). The 'ground-state energy function' is then $m_0(h)$.

A similar theory obviously also applies to links.

5. I am tempted to say at this point: cf. $E = mc^2$.

Notes:

There are obvious points of comparison with the theory of 'ideal' or 'tight' knots, based on minimising length for fixed circular cross-section. The main differences are as follows:

1. The cross-section of the tube T is not constrained to be circular in magnetic relaxation, and it will certainly deform during the final stage of relaxation; it can however be artificially constrained to remain circular, and the relaxation will not then proceed so far: $\bar{m}(h) > m(h)$, where $\bar{m}(h)$ applies to constrained relaxation.

2. More importantly, there is no counterpart of the helicity parameter h in the tight-knot scenario, in which only a single number, the minimum length for unit cross-section, is sought. This may correspond to the 'zero-framing' situation $h = 0$ in magnetic relaxation; but a precise correspondence has not been established.

3. The function $m(h)$ is not symmetric about $h = 0$, except for achiral knots. If $m(h)$ is the energy function for a chiral knot K, then the energy function of its mirror image K^* is $m(-h)$.

4. The constrained ground-state energy function $\bar{m}_0(h)$ has been computed by Chui & Moffatt (1995) for the first few torus knots.

3.5. *Reconnection*

The fact that tangential discontinuities (current sheets in the magnetostatic equilibria, vortex sheets in the analogous Euler flows) appear in the equilibrium states means that reconnections (with associated change of topology) are inevitable in real (as opposed to ideal) fluids. Such reconnection is diffusive in character, and involves dissipation of energy.

Reconnection is usually a consequence of the approach of two oppositely directed flux tubes (whether magnetic flux or flux of vorticity). In the case of vortex tubes, they may be driven together by the velocity induced (via Biot-Savart) by remote vorticity and/or by the approaching vortices themselves.

For diffusive reconnection of two initially skewed 'Burgers vortices', neglecting 'vortex-vortex' interaction, see Kimura & Moffatt (2014); The first sign of incipient reconnection is a 'bridge' connecting the approaching vortices, which become more and more nearly anti-parallel. Writhe helicity is destroyed during the resulting purely diffusive reconnection process, and is not compensated by twist helicity.

An alternative approach (Kimura & Moffatt 2017) starts with a concentrated vortex filament in the form of a figure-of-eight and uses just the Biot-Savart law to demonstrate the approach to a singularity at the point of nearest approach. The minimum separation D_m decreases like $(t_c - t)^{1/2}$ ('Leray scaling') and the velocity blows up like $(t_c - t)^{-1/2}$. The model fails to take account of deformation of the vortex cores which is known to avert the singularity. The model is therefore inadequate to describe the final stage of reconnection in a classical fluid, but may be better at describing reconnection of quantum vortices in liquid helium, which have δ-function cores that can't be deformed.

4. Topological jumps of minimum area soap-films

I will not attempt to summarise this topic here; I merely refer to a recent publication (Moffatt et al. 2016) that can be accessed at <http://www.damtp.cam.ac.uk/user/hkm2/PDFs/Moffatt2016a.pdf>.

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